11/7 Recitation Solutions

1. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ -1 & 0 & 0 \end{bmatrix}$.

   (a) Diagonalize A; that is, find matrices P and D satisfying $A = PDP^{-1}$. Use the Julia code in the **Julia12Diagonalization** notebook to check your work.

   Using a cofactor expansion along the first row, we compute:

   $$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ -1 & 1-\lambda & -1 \\ -1 & 0 & -\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ 0 & -\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)(-\lambda)$$

   Now the eigenvalues are the solutions to $-\lambda(1-\lambda)^2 = 0$, which are $\lambda = 0, 1$. We now determine the eigenspaces.

   $\underline{E_0(A)}$: We must determine $N(A - 0I) = N(A)$, which we do by solving the system $\overline{A\mathbf{x} = \mathbf{0}}$. Reducing A to RREF yields

   $$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ -1 & 0 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

   Hence x_3 is the free variable and we have the general solution

   $$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_3 \\ x_3 \end{bmatrix}$$

   Choosing $x_3 = 1$ gives the eigenvector $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

   $\underline{E_1(A)}$: We must determine $N(A - I)$, which we do by solving the system $(A - I)\mathbf{x} = \overline{\mathbf{0}}$. Reducing A to RREF yields

   $$A - I = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

   Hence x_2, x_3 are the free variables and we have the general solution

   $$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Choosing $x_2 = 0$ and $x_3 = 1$ gives the eigenvector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$; choosing $x_2 = 1$ and $x_3 = 0$ gives the eigenvector $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Let $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Then $A = PDP^{-1}$.

Julia Code :

```
A = [1 0 0 ; -1 1 -1 ; -1 0 0]
valA,vecA = eigen(A)
D = Diagonal(valA)
vecA * D * inv(vecA)
```

(b) Find a formula for A^k

Since $D^k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 0^k & 0 & 0 \\ 0 & 1^k & 0 \\ 0 & 0 & 1^k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D$, we have that

$$A^k = (PDP^{-1})^k = PD^kP^{-1} = PDP^{-1} = A!$$

2. Recall that we used the fact that A and A^T have the same eigenvalues in the proof that Markov matrices have an eigenvalue of 1. Let's prove this.

(a) Show that A and A^T have the same characteristic polynomial; that is, show $\det(A - \lambda I) = \det(A^T - \lambda I)$.

$\det(A - \lambda I) = \det((A - \lambda I)^T) = \det(A^T - \lambda I^T) = \det(A^T - \lambda I)$.

(b) Deduce that A and A^T have the same eigenvalues.

The eigenvalues of A are the solutions to $\det(A - \lambda I) = 0$ and the eigenvalues of A^T are the solutions to $\det(A^T - \lambda I) = 0$. Since these equations are the same, they have the same solutions. Hence A and A^T have the same eigenvalues.

3. True/False. If a statement is true, prove it. If it is false, provide a counterexample.

(a) Similar matrices have the same determinant.

True. Let $A = PBP^{-1}$. Then

$$\det A = \det(PBP^{-1}) = (\det P)(\det B)(\det P^{-1}) = (\det P)(\det B)(\frac{1}{\det P}) = \det B.$$

(b) If A is not invertible, then it is not diagonalizable.

False. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible since it only has one pivot. However, it is diagonalizable since A is already diagonal. Setting $P = I$, we have that $A = IAI^{-1}$.

4. Complete Problem 1 in the **Julia12Diagonalization** notebook.

(a) Show that M is a regular Markov matrix.

First note that the entries of M are nonnegative and the entries of each column sum to 1; hence M is a Markov matrix. It is regular, since M^2 has all positive entries, as the following code shows:

Code:
```
M=[.2 0 .2; .8 .1 0; 0 .9 .8]
M^2
```

(b) Find the eigenvalues of M; verify that it has complex eigenvalues.

Code:
```
valM,vecM = eigen(M)
```

The above output shows that M has 1 as an eigenvalue, along with two complex eigenvalues.

(c) Find the steady state vector $\mathbf{q}$ of M.

Since the third entry of valM is the eigenvalue 1, the third column $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ of vecM is an eigenvector corresponding to 1. Since $\mathbf{q}$ is an eigenvector corresponding to 1, it is a multiple of $\mathbf{v}$. Let $\mathbf{q} = c\mathbf{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ cv_3 \end{bmatrix}$. Then $cv_1 + cv_2 + cv_3 = 1$ or

$c = 1/(v_1 + v_2 + v_3)$. Then $\mathbf{q} = \frac{1}{v_1+v_2+v_3}\mathbf{v}$. We can find this in Julia with the following code:

Code:

```
q = (1/(vecM[1,3]+vecM[2,3]+vecM[3,3]))*vecM[:,3]
```

We obtain (approximately) $\begin{bmatrix} .1698 \\ .1509 \\ .6792 \end{bmatrix}$.

(d) Let $\mathbf{x}_0 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3$, where $a_1 + a_2 + a_3 = 1$. Symbolically compute $M^k\mathbf{x}_0$ for some large value of k; verify that you obtain a vector close to $\mathbf{q}$.

Compute $M^{20}\mathbf{x}_0$ symbolically in Julia:

Code:

```
x = [symbols("a$i") for i=1:3]
M^20*x
```

This yields a vector approximately equal to $\begin{bmatrix} .1698(a_1 + a_2 + a_3) \\ .1509(a_1 + a_2 + a_3) \\ .6792(a_1 + a_2 + a_3) \end{bmatrix} = \begin{bmatrix} .1698 \\ .1509 \\ .6792 \end{bmatrix}$, which is close to $\mathbf{q}$.

5. Do the coding challenges in the **Julia12Diagonalization** notebook.